Math 210C Lecture 22 Notes

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1 Simple Artinian Rings, the Artin-Wedderburn Theorem, Idempotents, and Modules of Semisimple Rings

1.1 Simple artinian rings

Let R be a nonzero ring. Recall that R is semisimple if it is semisimple as a left R-module (direct sum of simple R-modules). Last time, we showed that if R is semisimple, then R is a finite direct sum of its minimal left ideals. So R is left artinian (i.e. it satisfies the descending chain condition on left ideals).

Proposition 1.1. *A left artinian simple ring is semisimple.*

Remark 1.1. You might think "of course a simple ring should be semisimple!" But simple was defined with respect to 2-sided proper ideals, and semisimple was defined with respect to 1 sided ideals.

Proof. Let R be left artinian and simple. Then we have $R \supsetneq J_1 \supsetneq J_2 \supsetneq \cdots$, with ideals J_i, which stops at a minimal left ideal. Set $M = \sum N_i$, where N_i are the sumple (minimal) left ideals. If $N \subseteq R$ is simple and $r \in R$, then we get a surjection of left R-modules $N \to Nr$. Since N is simple, the kernel must be 0 or N. So $N_r \cong N$, or $Nr = 0$. So $Nr \subseteq M$. This is true for all N, so $Mr \subseteq M$. So M is a 2-sided ideal. R is simple, and $M \neq 0$, so $M = R$.

By a lemma from last time, $R = \sum_{i=1}^k N_i$, where N_i is a minimal left ideal (choose k minimally). To show that R is a direct sum, we have that for $M_i = \sum_{j \neq i} N_j$, $M_\cap N_i = 0$ or N_i. But if it is N_i we contradict minimality. So we must have $M_\cap N_i = 0$. By Schur's lemma, all N_i occur. Thus, $R = \bigoplus_{i=1}^k N_i$. $\square$

1.2 The Artin-Wedderburn theorem

Theorem 1.1 (Artin-Wedderburn[1]). *A nonzero ring is semisimple if and only if it is isomorphic to a finite product of matrix algebras over division rings.*

[1] Wedderburn is pronounced with a w sound, not a v. He was Scottish, although his name looks German.

Proof. ($\Longrightarrow$): We have $R^{\mathrm{op}} \to \mathrm{End}_R(R)$ given by $r \mapsto (s \mapsto sr)$. If R is semisimple, then $R = N_1^{n_1} \oplus \cdots \oplus N_k^{n_k}$, where N_i are the distinct simple R-modules. Now $R^{\mathrm{op}} \cong \mathrm{End}_R(N_1^{n_1} \oplus \cdots \oplus N_k^{n_k}) \cong \prod_{i=1}^k M_{n_i}(D_i^{\mathrm{op}})$. By Schur's lemma, $D_i = \mathrm{End}_R(N_i)$ is a division ring.

($\Longleftarrow$): If $R = \prod_{i=1}^k M_{n_i}(E_i)$, where E_i is a division ring, the simple left R-modules, $N_i = E_i^{n_i}$ for some i, are sets of column vectors. Then $R \cong N_1^{n_1} \oplus \cdots \oplus N_k^{n_k}$. □

Corollary 1.1. *R is left artinian and simple if and only if $R \cong M_n(D)$, where D is a division ring.*

Corollary 1.2. *R is semisimple if and only if R is isomorphic to a finite direct product of left artinian, simple rings.*

Corollary 1.3 (Wedderburn)**.** *An algebra over a field F is semisimple if and only if it is isomorphic to a product of simple, finite dimensional F-algebras. A finite dimensional F-algebra is simple if and only if it is isomorphic to $M_n(D)$ for $n \geq 1$, where D is a finite dimensional F-division algebra.*

Definition 1.1. An F-algebra is **central simple** if it is simple and F is its center.

Proposition 1.2. *If D is a finite dimensional division algebra over an algebraically closed field F, then $D = F$.*

Proof. Let $\gamma \in D$. Then γ commutes with F, so $F(\gamma)$ is a field extension of F. Since D is finite dimensional over F, $F(\gamma)/F$ is algebraic. So $F(\gamma) = F$, and we get $\gamma \in F$. □

Corollary 1.4. *Any finite dimensional semisimple F-algebra with F algebraically closed is isomorphic to $\prod_{i=1}^k M_{n_i}(F)$ for some n_i, k.*

Corollary 1.5. *A commutative semisimple algebra over a field F is a finite product of finite field extensions of F.*

1.3 Idempotents

Definition 1.2. An element $e \in R$ is **idempotent** if $e^2 = e$.

Definition 1.3. Idempotents $e, f \in R$ are **orthogonal** if $ef = fe = 0$.

If e, f are orthogonal, then $(e+f)(e+f) = e^2 + f^2 = e + f$, so $e + f$ is also idempotent.

Definition 1.4. An idempotent e is **primitive** if eR is not a product of to two subrings of R.

Lemma 1.1. *$R \cong \prod_{i=1}^k R_i$, where R_i are rings, if and only if there exist mutually orthogonal $e_1, \ldots, e_k \in Z(R)$ with $e_1 + \cdots + e_k = 1$.*

Proof. ($\implies$): Suppose $R \cong \prod_i R_i$. Then let e_i be the identity of R_i.

($\impliedby$): Let $e_1 + \cdots + e_k = 1$, where $e_i \in Z(R)$ are mutually orthogonal idempotents. $\square$

Such idempotents are called **primitive central**.

Example 1.1. The ring $\prod_{i=1}^k M_{n_i}(D_i)$ has $e_i = \mathrm{id}_{M_{n_i}(D_i)} = I_{n_i}$.

1.4 Modules of semisimple rings

Lemma 1.2. *If M is a simple module over a semisimple ring, then M is a direct summand of R.*

Proof. Write $R = \prod_{i=1}^k M_{n_i}(D_i)$. This has central idempotents $e_i = \mathrm{id}_{M_{n_i}(D_i)}$. M is simple, so since $M \cong \bigoplus_{i=1}^k e_i M$, we get that $e_i M = M$ for some i. Then $M_{n_i}(D_i)\ toM$ sending $1 \mapsto m \neq 0$ is a surjection. SO $M_{n_i}(D_i)$ mod the direct sum of all but one colimns is isomorphic to M. But then $M_{n_i}(D_i)$ is isomorphic to a direct sum of columns, so the surjection $M_n(D_i) \to M$ is split. So M injects into R as a summand. $\square$

Theorem 1.2. *The following are equivalent:*

1. *R is semisimple.*
2. *Every R-module is semisimple.*
3. *Every R-module is projective.*
4. *Every R-module is injective.*

Proof. (2) $\implies$ (1): This is a special case.

(2) $\implies$ (3): Simple modules are summands of R by the lemma, so they are projective. So semisimple modules are projective.

(3) $\iff$ (4): Suppose every module is injective. Then look at

$$0 \longrightarrow \ker(\pi) \longrightarrow R \xrightarrow{\pi} P \longrightarrow 0$$

Then $\ker(\pi)$ is injective, so $\ker(\pi) \to P$ splits. Then $R \to P$ splits. The same works the other way. $\square$

We will finish the proof next time.